Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

CERTAIN CASES OF EXTRANEOUS ROOTS*

ANDREW S. HEGEMAN
Central High School, Newark, N. J.

Sir Oliver Lodge has somewhere referred to the equation as the ''most serious thing in mathematics.'' Certainly the solution of equations involves many serious difficulties in the teaching of mathematics. One such difficulty is the matter of so-called ''extraneous roots.''

An ''extraneous root'' may be defined as ''a value obtained for an unknown in the solution of an equation which is not a root of the equation.'' It may be noted first that authors of textbooks usually fail to give a definition of extraneous roots; second, that the definition given above is faulty, in that it leaves out the qualifying phrase ''by correct processes.''

What constitutes ''correct processes'' in the solution of equations—in particular, in the solution of an equation in one unknown quantity?

The processes which, as a rule, are accepted as correct are:

(a) Adding equals to equals.
(b) Subtracting equals from equals.
(c) Multiplying equals by equals.
(d) Dividing equals by equals.
(e) Extracting equal roots of equals.

These processes are usually called ''axiomatic'' after the usage of Euclid in geometry. But it should be noted at once that Euclid nowhere makes any reference to anything resembling the last two processes above mentioned (for the obvious reason that in dealing with lines, angles, and areas, he treats them as if they were independent of number) and also that algebraic solution and geometric demonstration are not the same thing. In particular is it to be noted that the logic of an algebraic solution should hold when the steps are taken in reverse sequence, and that this is not necessarily true of a geometric demonstration.

This sets a certain limit upon the process of multiplying equals by equals. It is perfectly clear that dividing equals by equals is not a valid process if the divisor is zero. It follows that multiplying equals by equals is not a valid process in the algebraic (reversible) argument, when the multiplier is zero.

---

* Papers read before the Association of Mathematics Teachers of New Jersey, November 1, 1919, and October 30, 1920.

For example, let us take the false equation

$$x + 7 = x + 5$$

If both sides are multiplied by $x - 3$, we have

$$x^2 + 4x - 21 = x^2 + 2x - 15$$

whence $\qquad x = 3$

which does not check the original equation, nor will any other value of x check it. The difficulty is explained by the fact that both sides of the false equation have been multiplied by zero.

Whereas, in the equation

$$x^2 - 5x = 7x - 35$$

if both members are divided by $x - 5$, we have

$$x = 7$$

which *does* check, in spite of the fact that both members have been divided by zero in the form $x - 5$, since $x = 5$ will also check the equation.

This consideration would seem to contravene the familiar process of "clearing of fractions," when the L. C. D. contains the unknown quantity. As a matter of fact equations are often found in textbooks which, when treated in this fashion, yield results which are not roots at all.

For example:

$$\frac{x+3}{x^2-1} + \frac{3-x}{x-x^2} + \frac{x+2}{x^2+x} = 0.\text{''}$$

Multiplying by the L. C. D. $x(x+1)(x-1)$ we obtain

$$x^2 + 3x + x^2 - 2x + 3 + x^2 + x - 2 = 0$$

$$3x^2 + 2x - 5 = 0$$

Factoring $\qquad (3x + 5)(x - 1) = 0$

Whence $\qquad x = -\tfrac{5}{3} \text{ or } x = 1$

The value $-\frac{5}{3}$ gives a satisfactory check, but the value 1 gives the following:

$$\tfrac{4}{0}+\tfrac{2}{0}+\tfrac{3}{2}=0 \text{ or “} \infty+\infty+\tfrac{3}{2}=0\text{”}$$

Whatever meaning may be given to ∞, it is not likely to apply in the case of this result. In the best circles it would be said that there is no division by zero at all possible.

Consider also the equation

$$\frac{3}{2x+1}=\frac{4}{4x^2-1}+\frac{2}{1-2x}$$

Multiplying by the L. C. D. $(2x+1)(2x-1)$ we obtain

$$6x-3=4-4x-2$$
$$10x=5$$
$$x=1/2$$

$$\text{Checking } \tfrac{3}{2}=\tfrac{4}{0}+\tfrac{2}{0} \text{ or “}\tfrac{3}{2}=\infty+\infty\text{” (?)}$$

This means that the equation as given has no root at all. Since every equation must have a root if it has any meaning at all, it follows that the example given is not an equation, but is a falsehood in the form of an equation.

The difficulty, of course, lies in the fact that in general it is impossible to tell, when “clearing an equation of fractions,” whether or not one is multiplying by zero, that is, by an expression which equals zero for the same value which satisfies the equation itself.

As a matter of fact, any rational fractional equation (meaning an equation in which the unknown appears in some denominator) can be reduced to the form $P/Q=R$ where P, Q and R are integral functions. In this case, $P=QR$ follows from the axioms only if Q does not equal zero; and any value of x which makes $Q=0$ cannot be called a root.

This raises the question as to the exact meaning of such an equation as $P/Q=R$.

If regard be had for the notion that division is merely the inverse of multiplication, this equation would mean that $P=QR$, in which, if $Q=0$, P must also equal 0, which may or may not afford a root according as this is or is not possible.

As for example in the second illustration above cited:

$$\frac{3}{2x+1}-\frac{4}{4x^2-1}-\frac{2}{1-2x}=0$$

$$\frac{6x-3-4+4x+2}{(2x+1)(2x-1)}=0$$

$$\frac{10x-5}{(2x+1)(2x-1)}=0$$

$$\frac{5(2x-1)}{(2x+1)(2x-1)}=0$$

$$\frac{5}{2x+1}=0$$

which must mean $5=0(2x+1)$, a palpable absurdity.

So far, the assumption has been that the fraction P/Q is in its lowest terms as regards the unknown; if it is not, then the equation may be written

$$\frac{TU}{TV}=R$$

which, if interpreted to mean

$$TU=RTV$$

leads to $$T(U-RV)=0$$

which has two or more sets of roots.

This resolves at once the difficulty arising from the form 0/0, unless $P/Q=R$ is understood to signify something other than the equation $P=QR$, and to have infinitely many roots by the simple device of transforming the fraction to "higher terms." This is, of course, absurd, from the point of view of general theory. That is, $P=QR$ and $PS=QRS$ are really different and not equivalent equations. Whence it follows that $P/Q=R$ and $PS/QS=R$ are likewise different equations.

The conclusion is briefly—that any fractional equation set for solution should be of such a character that in its resultant form $P/Q=R$, P/Q is irreducible, and that the "extraneous roots" so-called are not roots at all.

How should this affect the teaching of elementary algebra? I am not prepared to dogmatize about the matter, as teachers may in many instances prefer to continue with less exact methods for the sake of brevity, and textbook writers, in many cases, sidestep the whole question by care in the selection of exercises.

IRRATIONAL EQUATIONS

It follows at once that the familiar process of squaring both sides of an equation is not permissible, for the reason that the algebraic "axiom" "square roots of equals are equal" is not true, and this would be necessary to justify the step, as has been seen, since the axioms of algebraic demonstration apply *in reverse sequence* to the solutions of an equation.

For example, consider the false equation

$$\sqrt{x^2 - 13} = 1 - x$$

(This equation is false for the reason that there is no value of x which will make both members of the equation positive real numbers, as is necessary.)

Squaring both sides, we have

$$x^2 - 13 = 1 - 2x + x^2$$

whence $$x = 7$$

which does not check, of course.

Whereas extracting the square root of each member of a quadratic equation properly arranged affords a new equation which in turn affords a solution which *will* check, as is well known.

How, then, are radical equations to be solved?

Let it be noted at once that radical equations of the sort found in elementary textbooks, and on college entrance examination papers, are usually special in character, concocted for the confounding of the minds of young persons, and rarely found in advanced or applied mathematics. The typical form usually contains three quadratic radicals—two radicals only not constituting a sufficiently severe test for the aspiring neophyte. Occasionally four radicals appear, but this practice is regarded as dangerous.

Let us first consider a case with two— or even one radical.

$$\sqrt{x^2 + 3x - 3} = 3x - 4$$

Solving by the usual method of squaring both sides, we obtain

$$x^2 + 3x - 3 = 9x^2 - 24x + 16$$
$$8x^2 - 27x + 19 = 0$$
$$(8x - 19)(x - 1 = 0$$
$$x = 1, \text{ or } 2\tfrac{3}{8}$$

The value 1 satisfies the original equation but the value $2\frac{3}{8}$ does not.

The student naturally wants to be shown why a root obtained by processes which he has been taught to consider correct is not a root at all. He is usually told that it is an extraneous root. This explains nothing and he is just as puzzled as before.

Let us now consider a solution by the use of a rationalizing factor.

Writing the equation in the form

$$\sqrt{x^2 + 3x - 3} - (3x - 4) = 0,$$

the rationalizing factor for the left hand member then is

$$\sqrt{x^2 + 3x - 3} + (3x - 4)$$

The question now arises: Can this rationalizing factor equal zero? As it is not permissible to multiply both members of an equation by zero, it becomes necessary to consider the result obtained when we have equations derived from two different functions of x, $f(x)$ and $\Phi(x)$, and a third derived from their product $f(x)\cdot\Phi(x)$. It does not take elaborate argument to show that all the roots of $\Phi(x) = 0$ are roots of $\Phi(x)\cdot f(x) = 0$, and all the roots of $f(x) = 0$ are roots of $\Phi(x)\cdot f(x) = 0$. The question which remains unanswered is: How many roots, if any, has an *irrational* equation $\Phi(x) = 0$? I have thus far not found any general test for the number of roots of an irrational equation. But given such an equation $\Phi(x) = 0$, we may term a second equation $f(x) = 0$, which is so formed that $\Phi(x)\cdot f(x)$ is rational, an *auxiliary equation,* if $f(x) = 0$ has roots. If not, then $f(x)$ can be regarded merely as a multiplier of both aides of the original $\Phi(x) = 0$.

Therefore returning to our problem, if we multiply member for member, the original equation,

$$\sqrt{x^2 + 3x - 3} - (3x - 4) = 0$$

by a rationalizing factor in the form of the auxiliary equation

$$\sqrt{x^2+3x-3}+(3x-4)=0$$

we obtain

$$x^2+3x-3-9x^2+24x-16=0$$

$$-8x^2+27x-19=0$$

$$-1(8x-19)(x-1)=0$$

$$x=1 \text{ or } 2\tfrac{3}{8}$$

The value 1 is a root of the original equation while the value $2\frac{3}{8}$ is a root of the auxiliary equation. It is not at all difficult to explain the origin of the so-called extraneous root when the equation is solved in this way.

It may be interesting to note in passing that frequently both values are given in an answer book without a word of comment, while it sometimes happens that a note like the following is given: "Certain values obtained for x in a radical equation are not roots of the equation as it appears."

Let us take another example

$$\sqrt{3x+7}-x-3=0$$

The rationalizing factor for the left-hand member is

$$\sqrt{3x+7}+x+3$$

This is an expression which no value of x will make equal to zero. Therefore we multiply *both* sides of the equation by this rationalizing factor and obtain

$$3x+7-x^2-6x-9=0$$

$$x^2+3x+2=0$$

$$(x+1)(x+2)=0$$

$$x=-2 \text{ or } -1$$

Both the value -2 and the value -1 are roots of the original equation.

Let us now consider equations with more than one radical in them. For example

$$\sqrt{x+1}+1-\sqrt{2x}=0$$

A rationalizing factor for the left-hand member is

$$\sqrt{x+1}+1+\sqrt{2x}$$

As this expression cannot equal zero, we multiply both members by it, obtaining

$$x + 1 + 2\sqrt{x+1} + 1 - 2x = 0$$

whence

$$2\sqrt{x+1} - x + 2 = 0$$

Just as we would find it necessary to square a second time if we were squaring both sides, so we now need to use a second rationalizing factor: *i. e.*,

$$2\sqrt{x+1} + x - 2$$

As this expression may equal zero, we multiply member for member, the equation,

$$2\sqrt{x+1} - x + 2 = 0$$

by the auxiliary equation

$$2\sqrt{x+1} + x - 2 = 0$$

obtaining

$$4x + 4 - x^2 + 4x - 4 = 0$$

whence

$$-x^2 + 8x = 0$$

$$x = 0 \text{ or } 8$$

The value 8 is a root of the original equation and the value 0 is a root of the auxiliary equation used to afford a second rationalizing factor.

Let us take another case with three radicals in the equation:

$$2\sqrt{x+1} + \sqrt{x-2} - \sqrt{7x+4} = 0$$

A rationalizing factor for the left hand member is

$$2\sqrt{x+1} + \sqrt{x-2} + \sqrt{7x+4}$$

As this expression cannot equal zero, we multiply both members by it, obtaining

$$4x + 4 + 4\sqrt{x^2 - x - 2} + x - 2 - 7x - 4 = 0$$

$$4\sqrt{x^2 - x - 2} - 2x - 2 = 0$$

$$2\sqrt{x^2 - x - 2} - x - 1 = 0$$

The second rationalizing factor is

$$2\sqrt{x^2 - x - 2} + x + 1$$

As this expression may apparently equal zero, we multiply, member for member, the equation

$$2\sqrt{x^2 - x - 2} - x - 1 = 0$$

by the auxiliary equation

$$2\sqrt{x^2 - x - 2} + x + 1 = 0$$

obtaining

$$\begin{aligned} 4x^2 - 4x - 8 - x^2 - 2x - 1 &= 0 \\ 3x^2 - 6x - 9 &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x - 3)(x + 1) &= 0 \\ x &= 3 \text{ or } -1 \end{aligned}$$

The value 3 is a root of the original equation and -1 is a root of the auxiliary equation used as the second rationalizing factor.

Let us take one more case—that of a radical equation in the quadratic form.

$$3x^2 - 4x + 2 + 3\sqrt{3x^2 - 4x + 2} - 4 = 0$$

Factoring

$$(\sqrt{3x^2 - 4x + 2} + 4)(\sqrt{3x^2 - 4x + 2} - 1) = 0$$

Multiplying by the auxiliary equation

$$(\sqrt{3x^2 - 4x + 2} - 4)(\sqrt{3x^2 - 4x + 2} + 1) = 0$$

we obtain

$$\begin{aligned} (3x^2 - 4x + 2 - 16)(3x^2 - 4x + 2 - 1) &= 0 \\ (3x^2 - 4x - 14)(3x - 1)(x - 1) &= 0 \end{aligned}$$

whence

$$x = \frac{2 \pm \sqrt{46}}{3},\ \tfrac{1}{3}, \text{ or } 1$$

The values $\frac{1}{3}$ and 1 are roots of the original equation while $\frac{2 \pm \sqrt{46}}{3}$ are roots of the auxiliary equation used to afford a rationalizing factor.

I could continue to give examples taken from standard text books, but I think a sufficient number has been given to illustrate my contention that the logic of an algebraic solution should hold when the steps are taken in reverse sequence.